

Angus R. Simpson

Comparing the Q-equations and Todini-Pilati formulation for solving the water distribution system equations

Water Distribution Systems Analysis 2010, Proceedings of the 12th Annual Conference, WDSA 2010, Tucson, Arizona, United States, 12 -15 Sep 2010 / Kevin E. Lansey, Christopher Y. Choi, Avi Ostfeld and Ian L. Pepper (eds.): pp. 37-54

© 2010 American Society of Civil Engineers

[10.1061/41203\(425\)6](https://doi.org/10.1061/41203(425)6)

PERMISSIONS

<http://dx.doi.org/10.1061/9780784479018.ch03>

p. 12 – Posting papers on the Internet

Authors may post the final draft of their work on open, unrestricted Internet sites or deposit it in an institutional repository when the draft contains a link to the bibliographic record of the published version in the ASCE Civil Engineering Database. “Final draft” means the version submitted to ASCE after peer review and prior to copyediting or other ASCE production activities; it does not include the copyedited version, the page proof, or a PDF of the published version.

24 April, 2015

<http://hdl.handle.net/2440/80958>

Comparing the Q-equations and Todini-Pilati formulation for solving the water distribution system equations

by

Simpson, A.R.

Water Distribution System Analysis (WDSA) 2010

Citation:

Simpson, A.R. (2010). "Comparing the Q-equations and Todini-Pilati formulation for solving the water distribution system equations." *Water Distribution System Analysis (WDSA) 2010*, Tucson, Arizona, USA, 12-15 September.

For further information about this paper please email Angus Simpson at angus.simpson@adelaide.edu.au

COMPARING THE Q-EQUATIONS AND TODINI-PILATI FORMULATION FOR SOLVING THE WATER DISTRIBUTION SYSTEM EQUATIONS

Angus R. Simpson

School of Civil, Environmental and Mining Engineering, University of Adelaide,
Adelaide, South Australia, Australia
angus.simpson@adelaide.edu.au

Abstract

This paper compares the solution of the network equations based on two methods. In the first method considered, the Q -equations are solved. The only unknowns in the Q -equations are the flows in each of the pipes or links. The Q -equations are formulated by taking the continuity equations at each supply node. In addition, energy equations for the head losses around primary closed loops and required independent paths between fixed head nodes in the network are written in terms of the flows in each pipe contained within the loop or path. The second method considered here is the Todini and Pilati solution method. This approach deals with all unknown flows and heads simultaneously. The governing equations are the continuity equations and the head loss equations for each individual pipe in the network written in terms of the unknown heads at the ends of the pipe and the unknown flow in the pipe. A smart two-step algorithm was developed by Todini and Pilati (1988) to first solve for the heads and then to solve for the flows sequentially in an iterative process. A simple case study involving two reservoirs, two pipes and one junction is used to illustrate the sequence of iterates for both the Q -equations and the Todini-Pilati method. It turns out that the sequence of iterates for the flows in both methods is identical. Comparison of the results for the case study leads to some interesting insights.

Keywords:

water distribution systems, solving equations, flow equations, Global Gradient Algorithm, equation formulations

1. INTRODUCTION

The equations governing flow and head in a water distribution system are based on both continuity and head loss relationships for pipes. The form of the loss relationship for a pipe makes the solution procedure a non-linear one. Thus an iterative solver is required to be used to successively solve a set of linear approximations to the problem. Only pipes are considered in this paper, although it is easy enough to extend this to the inclusion of pumps and valves.

There are three types of governing equations for flow and head (HGL or pressure) in a network of pipes. These include:

- continuity of flow at each node
- head loss–flow relationships for each individual pipe
- conservation of energy for each closed simple loop and the required independent paths (each loop or path comprising a number of links). A path is taken between two reservoirs or nodes of fixed head—it is like an open loop.

There are a number of alternative formulations of the equations describing a water distribution system (Wood and Rayes (1981) – for the first three formulations; the fourth formulation has been introduced by the author; Todini and Pilati (1988) for the fifth formulation). The five formulations are:

- *flow equations* or *Q-equations* formulation in terms of unknown flows (Q s) in each link. Unknown heads at the nodes in the network can be computed after the flows have been determined.
- *head equations* or *H-equations* formulation in terms of unknown heads or HGLs (H s) at each node.
- *loop flow equations* formulation or *LF-equations* in terms of unknown loop flows (LF s)
- *flow and head equations* or *Q-H equations* formulation in terms of unknown flows and unknown heads.
- *Todini and Pilati Q-H equations* formulation in terms of unknown flows and unknown heads (Todini and Pilati 1988).

The *Q-equations* and the Todini-Pilati Q-H equations are considered in this paper. The basis for their solution are given in detail. The solution methodology for each formulation is demonstrated and compared on a two pipe example.

2. THE UNKNOWNNS

The general column vector of flows in pipes or other links (pumps, valves etc.) in a water distribution system is

$$\mathbf{q} = [Q_1, Q_2, \dots, Q_{NP}]^T \quad (1)$$

where

- Q_j = flow in pipe or link j (m^3/s or ft^3/s)
- NP = number of links

The general column vector of heads (excluding fixed head nodes such as reservoirs) is

$$\mathbf{h} = [H_1, H_2, \dots, H_{NJ}]^T \quad (2)$$

where

- H_i = nodal head at node i (m or ft) (excluding fixed head nodes such as reservoirs)
- NJ = number of junctions

3. DEFINITION OF TOPOLOGY MATRICES

3.1 The unknown head node incidence matrix \mathbf{A}_1

The unknown head node incidence matrix \mathbf{A}_1 stores information about the connectivity of the nodes and the links in the network. Elements in the unknown head node incidence matrix \mathbf{A}_1 are defined according to Table 1, that considers link (in row j of the matrix) and a node (in column i of the matrix). It is assumed that each link has a designated flow direction.

Table 1 Elements in the unknown head incidence matrix \mathbf{A}_1

Element	Value	Explanation
a_{ji}	-1	If the link (in row j of the \mathbf{A}_1 matrix) <i>enters</i> the node in the designated flow direction (in column i of the \mathbf{A}_1 matrix)
a_{ji}	0	If the link (in row j of the \mathbf{A}_1 matrix) <i>is not connected</i> to the node (in column i of the \mathbf{A}_1 matrix)
a_{ji}	1	If the link (in row j of the \mathbf{A}_1 matrix) <i>leaves</i> the node in the designated flow direction (in column i of the \mathbf{A}_1 matrix)

3.2 The fixed head node incidence matrix \mathbf{A}_2

The fixed head node incidence matrix \mathbf{A}_2 stores information about the connectivity of the fixed head nodes (often reservoirs) and the links in the network. The dimensions of the fixed head node incidence matrix \mathbf{A}_2 are $NP \times NF$ (number of links– NP rows by number of reservoirs or fixed head nodes– NF columns). Elements in the fixed head node incidence matrix \mathbf{A}_2 are defined according to Table 2, that considers a link (in row j of the matrix) and a fixed head node (in column f of the matrix).

Table 2 Elements in the fixed head node incidence matrix \mathbf{A}_2

Element	Value	Explanation
a_{jf}	-1	If the link (in row j of the \mathbf{A}_2 matrix) <i>enters</i> the fixed head node in the designated flow direction (in column f of the \mathbf{A}_2 matrix)
a_{jf}	0	If the link (in row j of the \mathbf{A}_2 matrix) <i>is not connected</i> to the fixed head node (in column f of the \mathbf{A}_2 matrix)
a_{jf}	1	If the link (in row j of the \mathbf{A}_2 matrix) <i>leaves</i> the fixed head node in the designated flow direction (in column f of the \mathbf{A}_2 matrix)

4. THE CONTINUITY AND PIPE HEAD LOSS EQUATIONS

4.1 Continuity at nodes

A general expression for the continuity of flow at node i with a demand designated as DM_i (also referred to as an offtake, delivery or outflow from a network) is given by Equation 3 as

$$\sum_{j=1}^{NP} [a_{ij} Q_j] + DM_i = 0 \quad \text{for nodes } i = 1, 2, \dots, NJ \quad (3)$$

where

- Q_j = flow in link j (m^3/s or ft^3/s)
- a_{ij} = an element of the transpose \mathbf{A}_1^T of the unknown head node incidence matrix (with values zero, -1 or +1 as per Table 1), thus only Q_j values for pipes connected to the node are included.

- DM_i = demand at the node i (m^3/s or ft^3/s)

The sign convention adopted is such that a flow away from the node is considered to be positive. Thus the demand DM_i at a node is also positive in sign.

4.2 Head loss for pipes

The general expression for the head loss in a pipe j located between nodes i and k is given by

$$h_{f_j} = r_j Q_j |Q_j|^{n-1} \quad \text{for } j = 1, \dots, NP \quad (4)$$

$$H_i - H_k = r_j Q_j |Q_j|^{n-1} \quad \text{for } j = 1, \dots, NP \quad (5)$$

where

- h_{f_j} = head loss in pipe j (m or ft)
- r_j = resistance factor for the pipe j depending on the head loss relationship assumed for the pipe (for example, Darcy–Weisbach (see Equation 6) or Hazen–Williams ($r_{HW_j} = 10.670 L_j / C_j^{1.852} D_j^{4.871}$) for SI units)
- Q_j = flow in the pipe (m^3/s or ft^3/s)
- n = exponent of the flow in the head loss equation (Darcy–Weisbach $n = 2$ or Hazen–Williams equation $n = 1.852$)
- H_i = nodal head or HGL at one end of the pipe at node i (m or ft)
- H_k = nodal head or HGL at the other end of the pipe at node k (m or ft)

The resistance factor for pipe j the Darcy-Weisbach head loss formulation is given by

$$r_{f_j} = \frac{8 f_j L_j}{\pi^2 g D_j^5} \quad (6)$$

5. THE Q-EQUATIONS

In the Q -equations formulation or flow equations, the NP link flows (including flows in pipes, pumps or valves) are treated as unknowns. A set of continuity equations at nodes are written for the network (see Equation 3). In addition, energy losses in closed simple loops and required independent paths are expressed in terms of flows (Ormsbee and Wood 1986).

5.1 The loop and path energy equations

The sum of the head losses around a loop, must of course, be zero. For systems with more than one reservoir, energy equations must also be written for an appropriate number of independent paths between reservoirs or nodes of fixed head. The sum of the head losses for a path between two reservoirs must be equal to the difference in the elevations between the reservoir water surfaces. Once the flow in each link has been determined, the heads or HGLs and thus the pressures at each node may be calculated. Energy equations around closed simple loops (also referred to as non-overlapping or natural loops) (Jeppson 1976) or between fixed head nodes along required independent paths in a network are nonlinear.

Upon traversing a closed simple loop the sum of pipe head losses around the loop must be zero. This is expressed as

$$\sum_{j \in L_s} h_{f_j} = 0 \quad \text{for each closed simple loop } s = 1, \dots, NL \quad (7)$$

where

- NL = number of simple closed loops in the network

The head loss equations for each of the pipes in a closed simple loop are now expressed in terms of unknown flows in the pipes by using Equation 4 as:

$$\sum_{j \in L_s} r_j Q_j |Q_j|^{n-1} = 0 \quad s = 1, 2, \dots, NL \quad (8)$$

where

- L_s = indices of pipes in a closed simple loop s

Now define the number of separate parts of the network as the variable NC . Usually NC is equal to one. Thus in addition to closed simple loop energy equations in Equation 8, required independent paths between reservoirs or fixed head nodes in the network must also be considered.

Consider a series of NPL_s pipes between two reservoirs designated as *reservoir m* and *reservoir q* . The head loss in the pipes between two reservoirs must be equal to the difference in elevation or HGL between the two reservoirs. Each path should be traversed in the same direction as the loop flow direction (usually clockwise). This may be expressed as

$$\sum_{j \in L_s} h_{f_j} = EL_m - EL_q \quad s = 1, 2, \dots, (NF - NC) \quad (9)$$

where

- EL_m = HGL or water surface elevation at the reservoir m (m or ft)
- EL_q = HGL or water surface elevation at the reservoir q (m or ft)
- NC = number of separate disconnected subnetworks
- $(NF - NC)$ = number of required independent paths

Again by substituting Equation 4 for each pipe in the path into Equation 9 gives

$$\sum_{j \in L_s} r_j Q_j |Q_j|^{n-1} = EL_m - EL_q \quad s = 1, 2, \dots, (NF - NC) \quad (10)$$

A disadvantage of the *Q-equations* formulation is that the loops and paths need to be found in order to formulate the energy equations.

5.2 The convergence stopping criterion

The stopping criterion for terminating the iterative solution procedure for both the Q -equations and the Todini and Pilati solution method in this paper is based on the maximum change in heads between one iteration and the next. The infinity-norm for the head changes for the $(k+1)^{\text{st}}$ iteration may be computed as:

$$\|\delta \mathbf{h}^{(k+1)}\|_{\infty} = \max_{i=1, \dots, NJ} (|\delta H_i^{(k+1)}|) \quad (11)$$

Thus the stopping test is such that the iterative solution process terminates when the following is true

$$\|\delta \mathbf{h}^{(k+1)}\|_{\infty} < \epsilon_{\text{stop}} \quad (12)$$

6. EXAMPLE FOR Q-EQUATIONS FORMULATION

Consider the solution of the flow equations for the network in Figure 1. Properties of the network are given in Table 3 and Table 4. Flows for each of the two pipes need to be solved for are Q_1 and Q_2 . The column vector of unknown flows at the k^{th} iteration for the two pipe network is

$$\mathbf{q}^{(k)} = [Q_1^{(k)}, Q_2^{(k)}]^T \quad (13)$$

Table 3 Pipe properties for the two pipe network

Pipe	Upstream node	Downstream node	D (mm)	L (m)	ϵ (mm)
1	2	1	300	1000	0.25
2	1	3	300	1000	0.25

Table 4 Node properties for the two pipe network

Node ID	Elevation (m)	Demand (L/s)
1	40.0	50
2	80.0	Reservoir
3	30.0	Reservoir

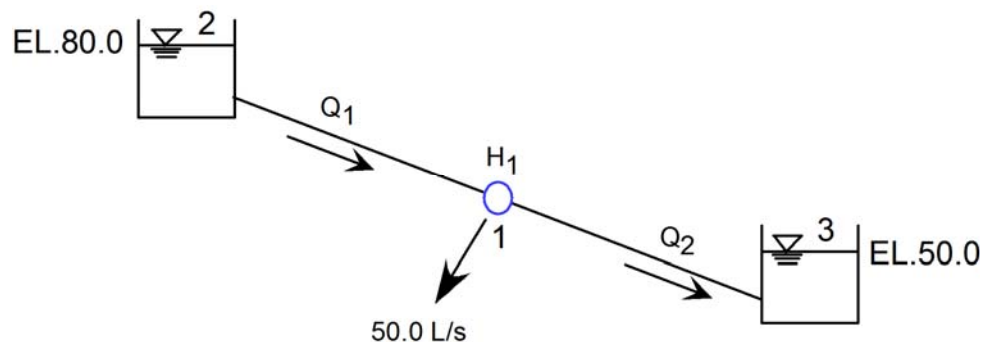


Figure 1 Unknown Q s and H to be solved for in the two pipe network

The *Q-equations* formulation of the governing equations for this two pipe network includes a continuity equation at node 1 and a required independent path equation between the two reservoirs at nodes 2 and 3 that includes pipes 1 and 2 (in a clockwise direction) as follows:

$$f_1(\mathbf{q}) = -Q_1 + Q_2 + DM_1 = 0 \quad \text{at node 1} \quad (14)$$

$$f_2(\mathbf{q}) = -r_1 Q_1 |Q_1| - r_2 Q_2 |Q_2| + (EL_2 - EL_3) = 0 \quad \text{along the path between reservoirs} \quad (15)$$

Based on the Newton solution technique for solving non-linear equations, the system to be solved iteratively for a sequence of $\delta \mathbf{q}^{(k+1)}$ values is:

$$\mathbf{J}(\mathbf{q}^{(k)}) \delta \mathbf{q}^{(k+1)} = -\mathbf{f}(\mathbf{q}^{(k)}) \quad (16)$$

The terms in the 2 by 2 Jacobian matrix at the k^{th} iteration can be determined by taking the appropriate partial derivatives of Equation 14 and Equation 15 as follows:

$$\mathbf{J}(\mathbf{q}^{(k)}) = \begin{bmatrix} \frac{\partial f_1(\mathbf{q}^{(k)})}{\partial Q_1} & \frac{\partial f_1(\mathbf{q}^{(k)})}{\partial Q_2} \\ \frac{\partial f_2(\mathbf{q}^{(k)})}{\partial Q_1} & \frac{\partial f_2(\mathbf{q}^{(k)})}{\partial Q_2} \end{bmatrix} = \begin{bmatrix} -1.0 & 1.0 \\ -2.0r_{f_1}|Q_1^{(k)}| & -2.0r_{f_2}|Q_2^{(k)}| \end{bmatrix} \quad (17)$$

ITERATION 1, *Q-equations*

The initial guesses for the flows in the two pipes in this example are based on a standard value of 1.0 fps (or 0.3048 m/s, Rossman 2000). Thus the vector of unknowns at iteration zero (0) is:

$$\mathbf{q}^{(0)} = \begin{bmatrix} 0.02155 \\ 0.02155 \end{bmatrix} \text{ m}^3/\text{s} \quad (18)$$

The Darcy-Weisbach friction factors (based on the Swamee and Jain equation) and corresponding *rf* values (from Equation 6) for the flows for pipes 1 and 2 in the two pipe network are given in Table 5.

Table 5 The *rf* values for initial flow guesses for *Q-equations* formulation

Pipe	V (m/s)*	D (m)	Re**	ϵ (mm)	<i>f</i>	<i>rf</i>
1	0.3048	0.3	90,828	0.25	0.02197	746.954
2	0.3048	0.3	90,828	0.25	0.02197	746.954

*Computed based on 1.0/3.28 for conversion of fps to m/s

** $\nu = 1.004 \times 10^{-6} \text{ m}^2/\text{s}$ at 20°C

The flows from Equation 18 may be substituted into the continuity and the path equations:

$$f_1(\mathbf{q}^{(0)}) = -Q_1 + Q_2 + 0.050 = -0.02155 + 0.02155 + 0.050 = 0.050 \text{ m}^3/\text{s} \quad (19)$$

$$f_2(\mathbf{q}^{(0)}) = -746.95Q_1|Q_1| - 746.95Q_2|Q_2| + 30.0 = -0.3469 - 0.3469 + 30 = 29.306 \text{ m} \quad (20)$$

Thus:

$$\mathbf{f}(\mathbf{q}^{(0)}) = \begin{bmatrix} 0.050 \\ 29.306 \end{bmatrix} \quad (21)$$

Thus the imbalance in the continuity equation is 50 L/s (Equation 19) while the imbalance in the head along the path is 29.306 m (Equation 20) and hence the flows are substantially underestimated by the initial guesses of 1.0 fps (0.3048 m/s), as not enough head loss is generated to offset the difference of 30.0 m between the two reservoirs.

The convergence check is based on Equation 12 and is expressed by the head change from iteration to iteration at node 1. Thus the head at node 1 needs to be computed as:

$$H_1^{(0)} = 80.0 - 746.95Q_1|Q_1| = 80.0 - 746.95(0.02155)|0.02155| = 80.0 - 0.3469 = 79.653 \text{ m} \quad (22)$$

The Jacobian matrix (Equation 17) for the first iteration is:

$$\mathbf{J}(\mathbf{q}^{(0)}) = \begin{bmatrix} -1.0 & 1.0 \\ -2.0r_{f_1}|Q_1^{(0)}| & -2.0r_{f_2}|Q_2^{(0)}| \end{bmatrix} = \begin{bmatrix} -1.0 & 1.0 \\ -32.195 & -32.195 \end{bmatrix} \quad (23)$$

The inverse of the Jacobian matrix is shown below for this simple two pipe network. Note that this would never be actually calculated in practice for a network of normal size. Instead a method such as a Gaussian lower upper decomposition (LUD) method or an alternative sparse matrix solution method would be used that avoids calculating the inverse explicitly.

$$\mathbf{J}^{-1}(\mathbf{q}^{(0)}) = \begin{bmatrix} -0.5 & -0.015531 \\ 0.5 & -0.015531 \end{bmatrix} \quad (24)$$

The flow corrections for the first iteration are calculated from Equation 16 based on Equation 21 and Equation 24 as:

$$\delta\mathbf{q}^{(1)} = -\mathbf{J}^{-1}(\mathbf{q}^{(0)})\mathbf{f}(\mathbf{q}^{(0)}) = -\begin{bmatrix} -0.5 & -0.015531 \\ 0.5 & -0.015531 \end{bmatrix} \begin{bmatrix} 0.050 \\ 29.306 \end{bmatrix} = \begin{bmatrix} 0.4801 \\ 0.4301 \end{bmatrix} \text{ m}^3/\text{s} \quad (25)$$

The new flows after the first iteration are:

$$\mathbf{q}^{(1)} = \mathbf{q}^{(0)} + \delta\mathbf{q}^{(1)} = \begin{bmatrix} 0.02155 \\ 0.02155 \end{bmatrix} + \begin{bmatrix} 0.4801 \\ 0.4301 \end{bmatrix} = \begin{bmatrix} 0.5017 \\ 0.4517 \end{bmatrix} \text{ m}^3/\text{s} \quad (26)$$

Note the flows have been increased in size by a factor of 20 to 25 times compared with the initial guesses. The flows for the first iteration from Equation 26 may now be substituted into the continuity equation of Equation 14 as follows:

$$f_1(\mathbf{q}^{(1)}) = -Q_1 + Q_2 + 0.050 = -0.5017 + 0.4517 + 0.050 = 0.000 \text{ m}^3/\text{s} \quad (27)$$

Note that the continuity equation is exactly satisfied after the first iteration. This remains the case for all subsequent iterations.

To check on convergence for iteration 1, use Equation 11 and Equation 12 for the head change infinity-norm as follows

$$H_1^{(1)} = 80.0 - 646.5Q_1|Q_1| = 80.0 - 646.5(0.5017)|0.5017| = 80 - 162.7 = -82.7 \text{ m} \quad (28)$$

$$\|\delta \mathbf{h}^{(k+1)}\|_{\infty} = |\delta H^{(1)}| = |H^{(1)} - H^{(0)}| = |-82.7 - 79.65| = 162.35 \text{ m} \quad (29)$$

Note in Equation 29 the resistance factors from Equation 6 have been updated based on the latest estimates of flows and friction factors.

The following is not satisfied

$$\|\delta \mathbf{h}^{(k+1)}\|_{\infty} < \varepsilon_{stop} \text{ where } \varepsilon_{stop} = 1.0 \times 10^{-6} \text{ m}$$

Thus the iterative process has not converged and needs to continue until convergence is obtained.

FINAL SOLUTION–ALL ITERATIONS, Q -equations

The iterations for the full solution based on the Q -equations formulation are shown in Table 6 and Table 7. Nine iterations are shown. A plot of the flows with each iteration during the solution process is shown in Figure 2. The stopping tolerance based on the infinity–norm of the head changes that was used in the computer simulation run for producing Table 6 was 1.0×10^{-6} m.

Table 6 Iteration sequence of flow corrections for the Q -equations formulation

Iteration	$\delta Q_1 \text{ m}^3/\text{s}$	$\delta Q_2 \text{ m}^3/\text{s}$
1	0.480142	0.430142
2	-0.214672	-0.214672
3	-0.088311	-0.088311
4	-0.023026	-0.023026
5	-0.002044	-0.002044
6	-4.34E-05	-4.34E-05
7	-6.38E-07	-6.38E-07
8	-9.29E-09	-9.29E-09
9	-1.35E-10	-1.35E-10

Table 7 Iteration sequence of flows for the Newton solution for the Q -equations formulation

Iteration	$Q_1 \text{ m}^3/\text{s}$	$Q_2 \text{ m}^3/\text{s}$	$H_1 \text{ (m)}$	$\ \delta H^{(k+1)}\ _{\infty}$
0	0.02155	0.02155	79.6531	
1	0.50169	0.45169	-82.7206	162.3740
2	0.28702	0.23702	26.3353	109.0560
3	0.19871	0.14871	54.0890	27.7538
4	0.17568	0.12568	59.6853	5.5963
5	0.17364	0.12364	60.1492	0.4640
6	0.17360	0.12360	60.1590	0.0098
7	0.17360	0.12360	60.1592	0.0001
8	0.17360	0.12360	60.1592	2.10E-06
9	0.17360	0.12360	60.1592	3.05E-08

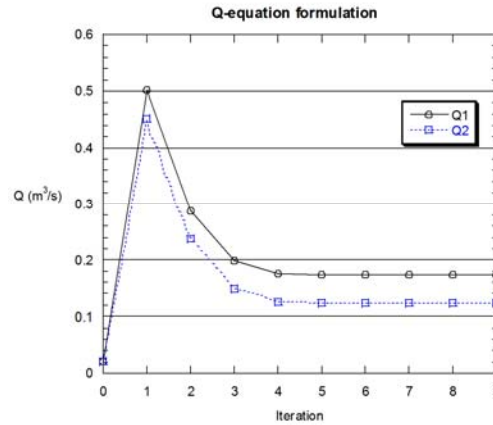


Figure 2. Convergence of flows for solution based on the *Q-equations* for the Newton method

7. TODINI AND PILATI Q-H FORMULATION

In the Todini and Pilati method both the heads and flows are solved for simultaneously in a sequential iterative process based on a smart partitioning of the problem. It is not necessary to define the loops for the Todini and Pilati formulation and this is a huge advantage of this technique compared to the *Q-equations* formulation. Only the link connectivity information is needed including the node numbers at the end of each link, the length, diameter and roughness of each pipe and the properties of each node (elevation and demand). The A_1 and A_2 matrices define the topology of the network. Simpson and Elhay (2009) gave a derivation of the Todini and Pilati equations based on an analytic inverse of a block form of the equations that followed the original paper. An alternative derivation is given below.

7.1 The continuity equations in matrix form—the Todini formulation

The matrix formulation of the *continuity equations* (Equation 3) for the Todini and Pilati formulation is

$$A_1^T \mathbf{q} + \mathbf{d}_m = \mathbf{0} \quad (30)$$

where

- A_1 = unknown head node incidence matrix
- \mathbf{q} = vector of unknown pipe flows (m³/s or ft³/s)
- \mathbf{d}_m = vector of known demands for each of the nodes (m³/s or ft³/s)

7.2 The pipe head loss equations in matrix form—the Todini formulation

Rearranging the pipe head loss equation (Equation 5) for the pipes not attached to reservoirs gives

$$-r_j Q_j |Q_j|^{n-1} + H_i - H_k = 0 \quad \text{for } j = 1, \dots, NP \text{ except reservoir nodes} \quad (31)$$

For pipes that are attached to a fixed head node reservoir with water surface elevation of EL_m the following equation applies

$$-r_j Q_j |Q_j|^{n-1} + EL_m - H_k = 0 \quad \text{for pipe } j \text{ for reservoir nodes} \quad (32)$$

Consider the first term $r_j Q_j |Q_j|^{n-1}$ on the left-hand side of Equation 31 and Equation 32. Introduce the following matrix form of \mathbf{G} (an NP by NP diagonal matrix) as follows

$$\mathbf{G} = \begin{bmatrix} r_1 |Q_1|^{n-1} & 0 & \dots & 0 & 0 \\ 0 & r_2 |Q_2|^{n-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & r_{NP-1} |Q_{NP-1}|^{n-1} & 0 \\ 0 & 0 & \dots & 0 & r_{NP} |Q_{NP}|^{n-1} \end{bmatrix} \quad (33)$$

Note that the \mathbf{G} is a positive diagonal matrix that is dependent on the flow vector \mathbf{q} . Another way to write \mathbf{G} is

$$\mathbf{G} = \text{diag}\{r_j |Q_j|^{n-1}\} \quad \text{for } j = 1, \dots, NP \quad (34)$$

Thus the resultant matrix form for the *head loss equations for the pipes* from Equation 31 and Equation 32 in a network is

$$-\mathbf{G}\mathbf{q} + \mathbf{A}_1 \mathbf{h} + \mathbf{A}_2 \mathbf{e}_L = \mathbf{0} \quad (35)$$

7.3 The recursive Todini and Pilati equations

A combined partitioned matrix formulation of the two sets of equations in Equation 30 and Equation 35 gives

$$\begin{bmatrix} \mathbf{G} & -\mathbf{A}_1 \\ -\mathbf{A}_1^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{h} \end{bmatrix} - \begin{bmatrix} \mathbf{A}_2 \mathbf{e}_L \\ \mathbf{d}_m \end{bmatrix} = \mathbf{0} \quad (36)$$

where

- \mathbf{G} = positive diagonal matrix of $r_j |Q_j|^{n-1}$ values
- n = exponent of the flow in the head loss equation (Darcy–Weisbach $n = 2$ or Hazen–Williams equation $n = 1.852$)
- \mathbf{q} = column vector of pipe flows (m^3/s or ft^3/s)
- \mathbf{A}_1 = unknown head node incidence matrix
- \mathbf{h} = column vector of heads for each of the nodes (m or ft)
- \mathbf{A}_2 = fixed node incidence matrix
- \mathbf{e}_L = reservoir elevations vector (m or ft)

Again the set of equations in Equation 36 is non-linear. The Newton method solution method is given as

$$\mathbf{J}(\mathbf{q}^{(k)}, \mathbf{h}^{(k)}) \begin{bmatrix} \delta \mathbf{q}^{(k+1)} \\ \delta \mathbf{h}^{(k+1)} \end{bmatrix} = -\mathbf{f}(\mathbf{q}^{(k)}, \mathbf{h}^{(k)}) \quad (37)$$

The Jacobian for this set of equations¹ is

$$\mathbf{J}(\mathbf{q}^{(k)}, \mathbf{h}^{(k)}) = \begin{bmatrix} n\mathbf{G}^{(k)} & | & -\mathbf{A}_1 \\ \hline -\mathbf{A}_1^T & | & \mathbf{0} \end{bmatrix} \quad (38)$$

This matrix is symmetric.

Now consider the function $\mathbf{f}(\mathbf{q}^{(k)}, \mathbf{h}^{(k)})$ term on the right-hand side from Equation 37

$$\begin{aligned} \mathbf{f}(\mathbf{q}^{(k)}, \mathbf{h}^{(k)}) &= \begin{bmatrix} f_1(\mathbf{q}^{(k)}, \mathbf{h}^{(k)}) \\ f_2(\mathbf{q}^{(k)}, \mathbf{h}^{(k)}) \end{bmatrix} = \begin{bmatrix} \mathbf{G}^{(k)} & | & -\mathbf{A}_1 \\ \hline -\mathbf{A}_1^T & | & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}^{(k)} \\ \mathbf{h}^{(k)} \end{bmatrix} - \begin{bmatrix} \mathbf{A}_2 \mathbf{e}_L \\ \mathbf{d}_m \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{G}\mathbf{q}^{(k)} - \mathbf{A}_1 \mathbf{h}^{(k)} - \mathbf{A}_2 \mathbf{e}_L \\ -\mathbf{A}_1^T \mathbf{q}^{(k)} - \mathbf{d}_m \end{bmatrix} \end{aligned} \quad (39)$$

Substituting Equation 38 and Equation 39 and introducing the unknowns into Equation 37 gives

$$\begin{bmatrix} n\mathbf{G}^{(k)} & | & -\mathbf{A}_1 \\ \hline -\mathbf{A}_1^T & | & \mathbf{0} \end{bmatrix} \begin{bmatrix} \delta \mathbf{q}^{(k+1)} \\ \delta \mathbf{h}^{(k+1)} \end{bmatrix} = - \begin{bmatrix} \mathbf{G}\mathbf{q}^{(k)} - \mathbf{A}_1 \mathbf{h}^{(k)} - \mathbf{A}_2 \mathbf{e}_L \\ -\mathbf{A}_1^T \mathbf{q}^{(k)} - \mathbf{d}_m \end{bmatrix} \quad (40)$$

where the superscripts k and $(k+1)$ refer to the iteration number. This is equivalent to Equation (9) in the Todini and Pilati (1988) paper. Note that matrices \mathbf{A}_1 , \mathbf{A}_2 and \mathbf{A}_1^T are all independent of the iteration number as too are the vectors \mathbf{e}_L and \mathbf{d}_m . Thus as a result these matrices and vectors do not require a iteration counter superscript. Consider the following expression

$$\begin{bmatrix} \delta \mathbf{q}^{(k+1)} \\ \delta \mathbf{h}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{q}^{(k+1)} - \mathbf{q}^{(k)} \\ \mathbf{h}^{(k+1)} - \mathbf{h}^{(k)} \end{bmatrix} \quad (41)$$

Substitute Equation 41 into Equation 40 to give

$$\begin{bmatrix} n\mathbf{G}^{(k)} & | & -\mathbf{A}_1 \\ \hline -\mathbf{A}_1^T & | & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}^{(k+1)} - \mathbf{q}^{(k)} \\ \mathbf{h}^{(k+1)} - \mathbf{h}^{(k)} \end{bmatrix} = - \begin{bmatrix} \mathbf{G}\mathbf{q}^{(k)} - \mathbf{A}_1 \mathbf{h}^{(k)} - \mathbf{A}_2 \mathbf{e}_L \\ -\mathbf{A}_1^T \mathbf{q}^{(k)} - \mathbf{d}_m \end{bmatrix} \quad (42)$$

1. Thanks go Dr. Sylvan Elhay, School of Computer Science, University of Adelaide for his elegant derivation of the Todini and Pilati equations

Thus it follows that

$$\begin{bmatrix} n\mathbf{G}^{(k)} & | & -\mathbf{A}_1 \\ \hline -\mathbf{A}_1^T & | & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}^{(k+1)} \\ \mathbf{h}^{(k+1)} \end{bmatrix} = \begin{bmatrix} n\mathbf{G}^{(k)} & | & -\mathbf{A}_1 \\ \hline -\mathbf{A}_1^T & | & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q}^{(k)} \\ \mathbf{h}^{(k)} \end{bmatrix} - \begin{bmatrix} \mathbf{G}\mathbf{q}^{(k)} - \mathbf{A}_1\mathbf{h}^{(k)} - \mathbf{A}_2\mathbf{e}_L \\ -\mathbf{A}_1^T\mathbf{q}^{(k)} - \mathbf{d}_m \end{bmatrix} \quad (43)$$

$$\begin{bmatrix} n\mathbf{G}\mathbf{q}^{(k+1)} - \mathbf{A}_1\mathbf{h}^{(k+1)} \\ -\mathbf{A}_1^T\mathbf{q}^{(k+1)} \end{bmatrix} = \begin{bmatrix} n\mathbf{G}\mathbf{q}^{(k)} - \mathbf{A}_1\mathbf{h}^{(k)} \\ -\mathbf{A}_1^T\mathbf{q}^{(k)} \end{bmatrix} - \begin{bmatrix} \mathbf{G}\mathbf{q}^{(k)} - \mathbf{A}_1\mathbf{h}^{(k)} - \mathbf{A}_2\mathbf{e}_L \\ -\mathbf{A}_1^T\mathbf{q}^{(k)} - \mathbf{d}_m \end{bmatrix} \quad (44)$$

or writing out the individual equations

$$n\mathbf{G}\mathbf{q}^{(k+1)} - \mathbf{A}_1\mathbf{h}^{(k+1)} = n\mathbf{G}\mathbf{q}^{(k)} - \mathbf{A}_1\mathbf{h}^{(k)} - (\mathbf{G}\mathbf{q}^{(k)} - \mathbf{A}_1\mathbf{h}^{(k)} - \mathbf{A}_2\mathbf{e}_L) \quad (45)$$

$$n\mathbf{G}\mathbf{q}^{(k+1)} - \mathbf{A}_1\mathbf{h}^{(k+1)} = (n-1)\mathbf{G}\mathbf{q}^{(k)} + \mathbf{A}_2\mathbf{e}_L \quad (46)$$

$$\mathbf{A}_1\mathbf{h}^{(k+1)} = n\mathbf{G}\mathbf{q}^{(k+1)} - (n-1)\mathbf{G}\mathbf{q}^{(k)} - \mathbf{A}_2\mathbf{e}_L \quad (47)$$

Multiply both sides of Equation 47 by $\mathbf{A}_1^T\mathbf{G}^{-1}$ gives

$$\mathbf{A}_1^T\mathbf{G}^{-1}[\mathbf{A}_1\mathbf{h}^{(k+1)}] = \mathbf{A}_1^T\mathbf{G}^{-1}[n\mathbf{G}\mathbf{q}^{(k+1)} - (n-1)(\mathbf{G}\mathbf{q}^{(k)} - \mathbf{A}_2\mathbf{e}_L)] \quad (48)$$

Now defining $\mathbf{U} = \mathbf{A}_1^T\mathbf{G}^{-1}\mathbf{A}_1$ gives

$$\mathbf{U}\mathbf{h}^{(k+1)} = n\mathbf{A}_1^T\mathbf{G}^{-1}\mathbf{G}\mathbf{q}^{(k+1)} - \mathbf{A}_1^T\mathbf{G}^{-1}[(n-1)\mathbf{G}\mathbf{q}^{(k)} + \mathbf{A}_2\mathbf{e}_L] \quad (49)$$

or

$$\mathbf{U}\mathbf{h}^{(k+1)} = n\mathbf{A}_1^T\mathbf{q}^{(k+1)} - (n-1)\mathbf{A}_1^T\mathbf{G}^{-1}\mathbf{G}\mathbf{q}^{(k)} - \mathbf{A}_1^T\mathbf{G}^{-1}\mathbf{A}_2\mathbf{e}_L \quad (50)$$

Rearranging from the continuity equation (Equation 30) gives

$$\mathbf{A}_1^T\mathbf{q}^{(k+1)} = -\mathbf{d}_m \quad (51)$$

which can be substituted into the first term on the right hand side of Equation 50 giving:

$$\mathbf{U}\mathbf{h}^{(k+1)} = -n\mathbf{d}_m + \mathbf{A}_1^T[(1-n)\mathbf{q}^{(k)} - \mathbf{G}^{-1}\mathbf{A}_2\mathbf{e}_L] \quad (52)$$

Now use Equation 47 to solve for the discharge

$$n\mathbf{G}\mathbf{q}^{(k+1)} = (n-1)\mathbf{G}\mathbf{q}^{(k)} + \mathbf{A}_2\mathbf{e}_L + \mathbf{A}_1\mathbf{h}^{(k+1)} \quad (53)$$

$$\mathbf{q}^{(k+1)} = \frac{1}{n}\mathbf{G}^{-1}\{(n-1)\mathbf{G}\mathbf{q}^{(k)} + \mathbf{A}_2\mathbf{e}_L + \mathbf{A}_1\mathbf{h}^{(k+1)}\} \quad (54)$$

Thus

$$\mathbf{q}^{(k+1)} = \frac{1}{n}\{(n-1)\mathbf{q}^{(k)} + \mathbf{G}^{-1}(\mathbf{A}_2\mathbf{e}_L + \mathbf{A}_1\mathbf{h}^{(k+1)})\} \quad (55)$$

Thus the summary of recursive Newton algorithm for the implementation of the Todini and Pilati algorithm is as follows:

$$\mathbf{U} = \mathbf{A}_1^T \mathbf{G}^{-1} \mathbf{A}_1 \quad (56)$$

$$\mathbf{h}^{(k+1)} = \mathbf{U}^{-1} \{-n\mathbf{d}_m + \mathbf{A}_1^T [(1-n)\mathbf{q}^{(k)} - \mathbf{G}^{-1} \mathbf{A}_2 \mathbf{e}_L]\} \quad (57)$$

$$\mathbf{q}^{(k+1)} = \frac{1}{n} \{(n-1)\mathbf{q}^{(k)} + \mathbf{G}^{-1} (\mathbf{A}_2 \mathbf{e}_L + \mathbf{A}_1 \mathbf{h}^{(k+1)})\} \quad (58)$$

In solving the set of non-linear equations for the heads (Equation 57), a linear approximation involves inverting a product of matrices called the Schur complement ($\mathbf{U} = \mathbf{A}_1^T \mathbf{G}^{-1} \mathbf{A}_1$) which is a square matrix that has the size of the number of unknown heads (NJ). This matrix needs to be effectively inverted to solve for the heads. Solving Equation 57 for the vector \mathbf{h} takes the majority of the computational effort in terms of dealing with the inverse of the Schur complement. Solving for the \mathbf{q} vector (Equation 58) requires only matrix multiplications as the inverse of the diagonal matrix \mathbf{G} is trivial. The inverse of \mathbf{G} at the k^{th} iteration is:

$$[\mathbf{G}^{(k)}]^{-1} = \begin{bmatrix} \frac{1}{r_1 |Q_1^{(k)}|^{n-1}} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{r_2 |Q_2^{(k)}|^{n-1}} & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{r_{NP-1} |Q_{NP-1}^{(k)}|^{n-1}} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{r_{NP} |Q_{NP}^{(k)}|^{n-1}} \end{bmatrix} \quad (59)$$

8. EXAMPLE FOR TODINI AND PILATI EQUATIONS FORMULATION

For Figure 1 for the two pipe network, the column vector of unknown heads and flows at the k^{th} iteration is

$$\mathbf{h}^{(k+1)} = [H_1^{(k+1)}] \text{ and } \mathbf{q}^{(k+1)} = \begin{bmatrix} Q_1^{(k+1)} \\ Q_2^{(k+1)} \end{bmatrix} \quad (60)$$

A column vector of initial estimates for the flows only in each pipe (initial estimates of the heads are not needed) is required to be made in order for the iterative solution process to begin:

$$\mathbf{q}^{(0)} = \begin{bmatrix} Q_1^{(0)} \\ Q_2^{(0)} \end{bmatrix} \quad (61)$$

In matrix form the unknown node incidence matrix \mathbf{A}_1 for the two pipe network [having 2 rows ($NP = 2$ for each of the two pipes) and 1 column ($NJ = 1$ for the one node)] is

$$\mathbf{A}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad (62)$$

The transpose of \mathbf{A}_1 is

$$\mathbf{A}_1^T = [-1 \quad 1] \quad (63)$$

The fixed node incidence matrix \mathbf{A}_2 for the two pipe network [having 2 rows ($NP = 2$ for the two pipes) and 2 columns ($NF = 2$ for the two fixed-head nodes)] is

$$\mathbf{A}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (64)$$

The demand vector [having 1 row ($NJ = 1$ for the one node) and one column] is

$$\mathbf{d}_m = [DM_1] = [0.050] \quad (65)$$

The fixed head elevation vector [having 2 rows ($NF = 2$ for the two reservoirs)] is

$$\mathbf{e}_L = \begin{bmatrix} EL_1 \\ EL_2 \end{bmatrix} = \begin{bmatrix} 80.0 \\ 50.0 \end{bmatrix} \text{ m} \quad (66)$$

ITERATION 1, Todini and Pilati Q–H equations formulation

The initial guesses for the flows in the two pipes in this example are based on a standard value of velocity of 1.0 fps (0.3048 m/s) and converted to a flow (same values as for the *Q-equations* formulation example). The vector of initial guesses of the unknowns at iteration zero (0) is:

$$\mathbf{q}^{(0)} = \begin{bmatrix} Q_1^{(0)} \\ Q_2^{(0)} \end{bmatrix} = \begin{bmatrix} 0.02155 \\ 0.02155 \end{bmatrix} \text{ m}^3/\text{s} \quad (67)$$

The friction factors (based on the Swamee and Jain equation) and corresponding r_f values for the flows in Equation 67 for pipes 1 and 2 in the two pipe network are given in Table 5 and were calculated for the *Q-equations* formulation. These are:

$$f^{(0)} = \begin{bmatrix} 0.02197 \\ 0.02197 \end{bmatrix} \text{ and } r_f^{(0)} = \begin{bmatrix} 746.95 \\ 746.95 \end{bmatrix} \quad (68)$$

The heads and flows to be solved successively at each iterative step will use Equation 57 and Equation 58.

Determining the Todini and Pilati h vector values for iteration 1:

The values from Equation 67 and Equation 68 based on estimates at the zeroth iteration may be substituted into the expression for the inverse of the \mathbf{G} matrix:

$$\mathbf{G}^{-1} = \begin{bmatrix} \frac{1}{r_1^{(0)}|Q_1^{(0)}|} & 0 \\ 0 & \frac{1}{r_2^{(0)}|Q_2^{(0)}|} \end{bmatrix} = \begin{bmatrix} \frac{1}{746.95|0.02155|} & 0 \\ 0 & \frac{1}{746.95|0.02155|} \end{bmatrix} = \begin{bmatrix} 0.062122 & 0 \\ 0 & 0.062122 \end{bmatrix} \quad (69)$$

Now solve for the unknown head at node 1 by using Equation 57 and Equation 56 as

$$\mathbf{h}^{(k+1)} = (\mathbf{A}_1^T \mathbf{G}^{-1} \mathbf{A}_1)^{-1} \{-n\mathbf{d}_m + \mathbf{A}_1^T [(1-n)\mathbf{q}^{(k)} - \mathbf{G}^{-1} \mathbf{A}_2 \mathbf{e}_L]\} \quad (70)$$

$$\mathbf{h}^{(1)} = \left(\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{r_1^{(0)}|Q_1^{(0)}|} & 0 \\ 0 & \frac{1}{r_2^{(0)}|Q_2^{(0)}|} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)^{-1} \left\{ -0.100 + \begin{bmatrix} -1 & 1 \end{bmatrix} \left(-\begin{bmatrix} Q_1^{(0)} \\ Q_2^{(0)} \end{bmatrix} - \begin{bmatrix} \frac{1}{r_1^{(0)}|Q_1^{(0)}|} & 0 \\ 0 & \frac{1}{r_2^{(0)}|Q_2^{(0)}|} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 80.0 \\ 50.0 \end{bmatrix} \right) \right\} \quad (71)$$

Substituting for \mathbf{G}^{-1} from Equation 69 at the zeroth iteration and $\mathbf{q}^{(0)}$ gives

$$\mathbf{h}^{(1)} = \left(\begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.062122 & 0 \\ 0 & 0.062122 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)^{-1} \left\{ -0.100 + \begin{bmatrix} -1 & 1 \end{bmatrix} \left(-\begin{bmatrix} 0.02155 \\ 0.02155 \end{bmatrix} - \begin{bmatrix} 0.062122 & 0 \\ 0 & 0.062122 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 80.0 \\ 50.0 \end{bmatrix} \right) \right\} \quad (72)$$

Thus

$$\mathbf{h}^{(1)} = 64.1949 \text{ m} \quad (73)$$

and

$$\left[H_1^{(1)} \right] = 64.1949 \text{ m} \quad (74)$$

Determining the Todini and Pilati q vector values for iteration 1:

The $\mathbf{q}^{(1)}$ value can now be obtained from Equation 58 of

$$\mathbf{q}^{(k+1)} = \frac{1}{n} \{ (n-1)\mathbf{q}^{(k)} + \mathbf{G}^{-1} (\mathbf{A}_2 \mathbf{e}_L + \mathbf{A}_1 \mathbf{h}^{(k+1)}) \} \quad (75)$$

$$\mathbf{q}^{(1)} = 0.5 \left\{ \begin{bmatrix} Q_1^{(0)} \\ Q_2^{(0)} \end{bmatrix} + \begin{bmatrix} \frac{1}{r_1^{(0)}|Q_1^{(0)}|} & 0 \\ 0 & \frac{1}{r_2^{(0)}|Q_2^{(0)}|} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 80.0 \\ 50.0 \end{bmatrix} + \begin{bmatrix} \frac{1}{r_1^{(0)}|Q_1^{(0)}|} & 0 \\ 0 & \frac{1}{r_2^{(0)}|Q_2^{(0)}|} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} H_1^{(1)} \end{bmatrix} \right\} \quad (76)$$

Now substituting into Equation 76 gives

$$\mathbf{q}^{(1)} = 0.5 \left\{ \begin{bmatrix} 0.02155 \\ 0.02155 \end{bmatrix} + \begin{bmatrix} 4.96978 \\ -3.1061 \end{bmatrix} + \begin{bmatrix} 0.062122 & 0 \\ 0 & 0.062122 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} 64.1949 \end{bmatrix} \right\} \quad (77)$$

Thus

$$\mathbf{q}^{(1)} = \begin{bmatrix} Q_1^{(1)} \\ Q_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0.50169 \\ 0.45169 \end{bmatrix} \text{ m}^3/\text{s} \quad (78)$$

FINAL SOLUTION–ALL ITERATIONS, Todini and Pilati Q–H equations

The iterations for the full solution based on the *Todini and Pilati Q–H equations* formulation are shown in Table 8. Eight iterations are shown. The plot of head versus iteration is shown in Figure 3. The stopping tolerance based on the infinity–norm of the head changes that was used in the computer simulation run for producing Table 8 was 1.0×10^{-6} m.

Table 8 Iteration sequence of flows and head for the *Todini and Pilati Q–H equations* formulation

Iteration	Q_1 m ³ /s	Q_2 m ³ /s	H_1 (m)	$\ \delta H^{(k+1)}\ _{\infty}^{**}$
0	0.02155	0.02155		
1	0.50169	0.45169	64.195	15.458
2	0.28701	0.23701	56.531	7.664
3	0.19869	0.14869	59.357	2.826
4	0.17566	0.12566	60.093	0.736
5	0.17361	0.12361	60.157	0.064
6	0.17357	0.12357	60.158	0.001
7	0.17357	0.12357	60.158	1.68E-05
8	0.17357	0.12357	60.158	2.44E-07

**Infinity-norm for head changes (m) used for convergence = $\max_{i=1, \dots, NJ} (|\delta H_i^{(k+1)}|)$

Note the values of pipe flows from the Todini-Pilati method are exactly the same as for the values at iteration 1 for the solution to the *Q-equations* formulation. Thus the plot of Figure 2 also applies to the *Todini and Pilati* method.

To solve the *Todini and Pilati Q–H equations* a Jacobian matrix only of size $NJ = 1$ had to be inverted to solve for the heads. For the *Q-equations* formulation a Jacobian matrix of size $NP = 2$ had to be inverted. An interesting feature is that the sequence of iterates of Q s for the *Q-equations* formulation is identical to the *Todini and Pilati Q–H equations* formulation iterates. An interesting difference between the two methods is the significantly different sequence of head values for node 1. The Todini and Pilati values are much closer to the final result even after one iteration than the *Q-equations* method values. The Todini and Pilati method takes one less iteration to converge than the *Q-equations* method.

9. CONCLUSIONS

This paper has compared two different solution methods for finding the heads and flows in a water distribution system. Solution methods for both the *Q-equations* formulation and the *Todini and Pilati H-Q equations* have been given. Both techniques have been applied to solving a two reservoir, two pipe, one node system. The Todini and Pilati method has to deal with a smaller number of unknowns (NJ - the number of unknown heads) in computing the inverse while the *Q-equations* has to invert a matrix that is NP – the number of unknown heads in size. It turns out that the sequence of Q-iterates is identical for both solution methods.

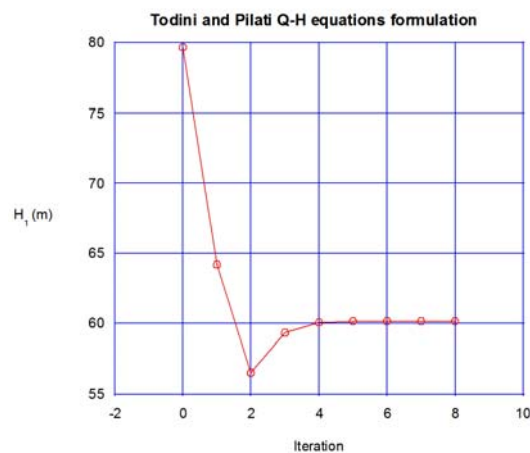


Figure 3 Convergence of head at node 1 for solution of the two pipe network using the *Todini and Pilati Q-H equations*

References

- Jeppson, R. (1976). *Analysis of Flow in Pipe Networks*, p. 164pp. Butterworth-Heinemann.
- Ormsbee, L. & Wood, D. (1986). 'Hydraulic design algorithms for pipe networks'. *Journal of Hydraulic Engineering* **112**(12):1195–1207.
- Rossman, L. A. (2000). *Epanet 2 users' manual*, USEPA, Cincinnati.
- Simpson, A.R. and Elhay, S. (2009). "A framework for alternative formulations of the pipe network equations" *11th Annual Symposium on Water Distribution Systems Analysis*, American Society of Civil Engineers, Kansas City, USA, 17-21 May.
- Todini, E., and Pilati, S. (1988). "A gradient method for the analysis of pipe networks." *Computer applications in water supply*, B. Coulbeck and C. H. Orr, eds., Wiley, London, 1–20.
- Wood D. & Rayes, A. (1981). 'Reliability of Algorithms for Pipe Network Analysis'. *Journal of the Hydraulic Division* **107**(HY10):1145–1161.